2022 BC #6
(no calculator)

(a)

Using the ratio test, we want to find all x such that

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{2(n+1)+1}\bullet\frac{2n+1}{x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\bullet\frac{2n+1}{x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^{2n}x^{3}}{2n+3}\bullet\frac{2n+1}{x^{2n}x^{1}}\right|=\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)\bullet\left|x^2\right|<1$$

$\left|x^2\right|<1 \;\Rightarrow\; -1<x<1$ and the radius of convergence $=1$

Testing the endpoints:

When $x=-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ When $x=1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$

Both are alternating series whose terms decrease in absolute value to 0 so they both converge.

In other words they are alternating series, $a_{n+1}<a_n$, and $\displaystyle\lim_{n\to\infty}\frac{1}{2n+1}=0$

So the interval of convergence of f is $\boxed{-1\le x\le 1}$.

(b)

$f\left(\frac{1}{2}\right)\approx\frac{1}{2}$ so we can say that this represents $P_1\left(\frac{1}{2}\right)$, the first degree

Taylor polynomial for the alternating series, $f(x)$ when $x=\frac{1}{2}$.

So, $\left|f(x)-\frac{1}{2}\right|=\left|f(x)-P_1\left(\frac{1}{2}\right)\right|$ is the error form for the alternating series.

Hence, the alternating series error bound is the first omitted term $\Rightarrow$

$$\left|f(x)-\frac{1}{2}\right|=\left|f(x)-P_1\left(\frac{1}{2}\right)\right|\le\left|\frac{-\left(\frac{1}{2}\right)^3}{3}\right|=\frac{1}{24}<\frac{1}{10}$$

(c)

$$f'(x)=\boxed{1-\frac{3x^2}{3}+\frac{5x^4}{5}-\frac{7x^6}{7}+\cdots+\frac{(2n+1)(-1)^n x^{2n}}{2n+1}+\cdots}$$

or $\quad =1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$

(d)

$$f'\left(\frac{1}{6}\right)\approx 1-\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^4-\left(\frac{1}{6}\right)^6+\cdots=\boxed{\frac{1}{1-\left(-\left(\frac{1}{6}\right)^2\right)}}\quad\text{or}\quad\frac{36}{37}$$

$\left(\text{a geometric series where } a_1=1 \text{ and } r=-\left(\frac{1}{6}\right)^2 \text{ so the sum}=\frac{a_1}{1-r}\right)$