2023 BC #6
(no calculator)

(a)

$$f^{(4)}(x) = \left(f'''(x)\right)' = -2x \bullet f''(x^2) \bullet 2x + f'(x^2) \bullet -2 = \boxed{-4x^2 \bullet f''(x^2) - 2 \bullet f'(x^2)}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = 2 + 3x + \frac{-f(0)}{2!}x^2 + \frac{-2(0)f'(0)}{3!}x^3 + \frac{-4(0)f''(0) - 2f'(0)}{4!}x^4$$

$$P_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{0 - 2 \bullet 3}{4!}x^4$$

$$P_4(x) = \boxed{2 + 3x - x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4} = 2 + 3x - x^2 - \frac{1}{4}x^4$$

(b)

$P_4(0.1)$ is used to approximate $f(0.1)$. It is given that $\left|f^{(5)}(x)\right| \le 15$ on the interval $[0, 0.5]$.

The error in this approximation using the Lagrange error bound is:

$$\left|f(0.1) - P_4(0.1)\right| \le \left|\frac{\underset{\max 0 \le x \le 0.1}{f^{(5)}(x)}}{5!}(0.1)^5\right| \le \frac{15}{120}\left(\frac{1}{10^5}\right) = \frac{1}{8}\left(\frac{1}{10^5}\right) < \frac{1}{10^5}$$

(c)

$g(0) = 4$

$g'(x) = e^x f(x) \;\rightarrow\; g'(0) = e^0 f(0) = 1 \bullet 2 = 2$

$g''(x) = e^x f'(x) + f(x)e^x \;\rightarrow\; g''(0) = e^0 f'(0) + f(0)e^0 = 3 + 2 = 5$

$$P_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2$$

$$\boxed{P_2(x) = 4 + 2x + \frac{5}{2}x^2}$$